Attitude Estimation on SO(3) with Unknown Input

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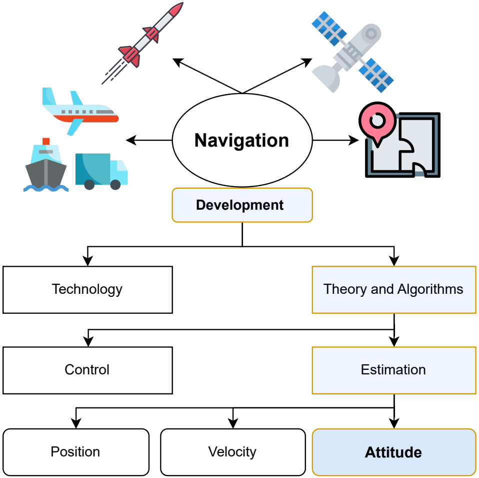
May 31, 2024

Background


Navigation
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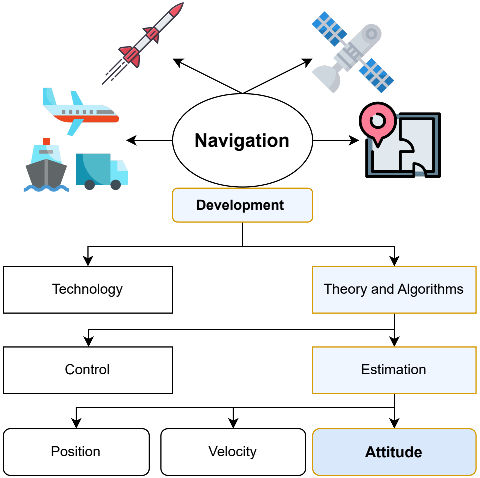
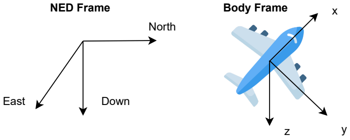
Background



Attitude is the way in which the rigid body is disposed in space.



The attitude state representation: Euler angles.

$$\phi \in [-\pi, \pi],\ \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}],\ \psi \in [-\pi, \pi]]$$

Quaternion.

$$\mathbf{q} \in \mathbb{R}^4 : \mathbf{q}^T\mathbf{q} = 1$$

Rotation matrices.

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times3} : \mathbf{R}\mathbf{R}^T = \mathbf{I}_3\ ,\ \det(\mathbf{R}) = 1\}$$

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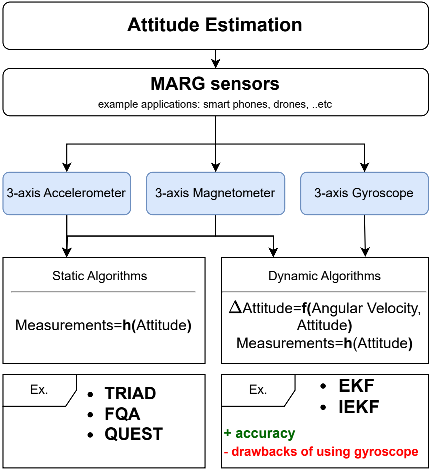

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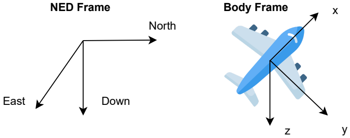

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Why not using Gyroscope

The drawbacks of using a gyroscope:

- Relatively high-power consumption [Liu et al., 2017].
- Bias and noise [Hiller et al., 2019].
- The need to know the noise covariance matrix.

Solutions for attitude estimation without using a gyroscope:

- Static algorithms. (TRIAD, QUEST [Shuster and Oh, 1981])
- Angular motion estimation algorithms. [Ciblak, 2007]

New approach

Our solution is to consider the angular velocity as an unknown input and hence avoid the use of a gyroscope for attitude estimation.

Problem statement

The discrete-time state-space model:

$$\mathbf{R}_k = \mathbf{f}(\mathbf{R}_{k-1}, \omega_{k-1}, \mathbf{w}_{k-1})$$

$$\mathbf{y}_k = \begin{pmatrix} \mathbf{a}_k^b \\ \mathbf{b}_k^b \end{pmatrix} = \begin{pmatrix} \mathbf{R}_k^{-1}\mathbf{g} \\ \mathbf{R}_k^{-1}\mathbf{m}_e \end{pmatrix} + \mathbf{v}_k^y$$

$$\omega_k, \mathbf{a}_k^b, \mathbf{b}_k^b, \mathbf{g}, \mathbf{m}_e, \mathbf{w}_k \in \mathbb{R}^3$$

$$\mathbf{y}_k, \mathbf{v}_k^y = \begin{pmatrix} \mathbf{v}_k^g \\ \mathbf{v}_k^m \end{pmatrix} \in \mathbb{R}^3$$

$$E(\mathbf{v}_k^y \mathbf{v}_k^{y\,T}) = \mathcal{R}_k, E(\mathbf{w}_k \mathbf{w}_k^T) = \mathcal{Q}_k$$

$$\mathbf{f} : SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow SO(3)$$

Our aim is to design a dynamic algorithm to estimate the attitude without using gyroscope measurements while considering them as unknown inputs.

The second contribution

Design UMV-SO(3) algorithm for state estimation on SO(3) with unknown input without having direct feedthrough to the output (measurement) function.

Filter design

For linear discrete-time systems with unknown input without direct feedthrough to the output, the four-step Kalman filter was introduced in [Gillijns and De Moor, 2007][1] for both state and unknown input estimation.

1. Gives a biased prediction of the state using the state dynamic model.
2. Estimate the unknown input using the biased predicted state and the measurements.
3. Gives unbiased prior estimation of the state using the unknown input estimate.
4. Corrects the prior state estimate using the measurements.

[1] S. Gillijns and B. De Moor. Unbiased minimum-variance input and state estimation for linear discrete-time systems. Automatica, 43(1):111–116, 2007.

Special Orthogonal Group SO(3)

Special Orthogonal Group SO(3):

$$SO(3) = \{\mathbf{R} \in \mathbb{R}^{3\times 3} : \mathbf{R}\mathbf{R}^T = \mathbf{I}_3 \ , \ \det(\mathbf{R}) = 1\}$$

The exponential map: $\exp_m : \mathbb{R}^3 \to SO(3)$

$$\exp_m(\xi) = \exp((\xi)_\times)$$

$$(\xi)_\times = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}_\times = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}$$

$$\exp_m(\xi) = \mathbf{I}_3 + \frac{\sin(\|\xi\|)}{\|\xi\|}(\xi)_\times + 2\frac{\sin(\|\xi\|/2)^2}{\|\xi\|^2}(\xi)_\times^2$$

Some properties:

$$\exp_m(\xi)^{-1} = \exp_m(-\xi)$$

$$(\xi_1)_\times \xi_2 = -(\xi_2)_\times \xi_1$$

Error representation:

$$\exp_m(\xi) = \hat{\mathbf{R}}^T\mathbf{R}$$

$$\mathbf{P} = \mathbf{E}(\xi\xi^T)$$

The dynamic equation:

$$\mathbf{R}_k = \mathbf{R}_{k-1}\exp_m(\omega_{k-1} + \mathbf{w}_{k-1})$$

Prediction

The dynamic equation:

$$\mathbf{R}_k = \mathbf{R}_{k-1} \exp(\omega_{k-1} + \mathbf{w}_{k-1})$$

The prediction:

$$\begin{aligned} \hat{\mathbf{R}}_{k|k-1} &= \hat{\mathbf{R}}_{k-1} \\ \exp(\xi_{k|k-1}) &= \hat{\mathbf{R}}_{k|k-1}^{-1} \mathbf{R}_k \\ &\vdots \\ \xi_{k|k-1} &= \xi_{k-1} + \omega_{k-1} + \mathbf{w}_{k-1} \\ E(\xi_{k|k-1}) &= \omega_{k-1} \end{aligned} \quad (1)$$

Prediction

The dynamic equation:

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The prediction:

$$\boxed{\hat{\mathbf{R}}_{k|k-1} = \hat{\mathbf{R}}_{k-1}} \tag{1}$$

$$\exp(\xi_{k|k-1}) = \hat{\mathbf{R}}_{k|k-1}^{-1} \mathbf{R}_k$$

$$\vdots$$

$$\xi_{k|k-1} = \xi_{k-1} + \omega_{k-1} + \mathbf{w}_{k-1}$$

$$E(\xi_{k|k-1}) = \omega_{k-1}$$

Angular velocity estimation

The output function:

$$\mathbf{y}_k = \begin{pmatrix} \mathbf{a}_k^b \\ \mathbf{b}_k^b \end{pmatrix} = \begin{pmatrix} \mathbf{R}_k^{-1}\mathbf{g} \\ \mathbf{R}_k^{-1}\mathbf{m}_e \end{pmatrix} + \mathbf{v}_k^y$$
$$= \mathbf{h}(\mathbf{R}_k) \qquad + \mathbf{v}_k^y$$

$$\mathbf{h}(\mathbf{R}_k) - \mathbf{h}(\hat{\mathbf{R}}_{k|k-1}) \approx \mathbf{H}_k \xi_{k|k-1}$$

$$\mathbf{H}_k = \begin{pmatrix} (\hat{\mathbf{R}}_{k|k-1}^{-1}\mathbf{g})_\times \\ (\hat{\mathbf{R}}_{k|k-1}^{-1}\mathbf{m}_e)_\times \end{pmatrix} \qquad (2)$$

The angular velocity estimation:

$$\tilde{\mathbf{y}}_k = \mathbf{y}_k - \mathbf{h}(\hat{\mathbf{R}}_{k|k-1})$$
$$\vdots$$
$$\tilde{\mathbf{y}}_k = \mathbf{H}_k \boldsymbol{\omega}_{k-1} + \mathbf{e}_k \qquad (3)$$

where $E(\mathbf{e}_k) = 0,\ Cov(\mathbf{e}_k) = \tilde{\mathcal{R}}_k$

$$\tilde{\mathcal{R}}_k = \mathbf{H}_k \mathbf{P}_{k-1} \mathbf{H}_k^T + \mathbf{H}_k \mathcal{Q}_{k-1} \mathbf{H}_k^T + \mathcal{R}_k \qquad (4)$$

$$\mathbf{M}_k = (\mathbf{H}^T \tilde{\mathcal{R}}_k^{-1} \mathbf{H})^{-1} \mathbf{H}^T \tilde{\mathcal{R}}_k^{-1} \qquad (5)$$
$$\hat{\omega}_k = \mathbf{M}_k(\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{R}}_{k|k-1})) \qquad (6)$$

Angular velocity estimation

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Unbiased estimation and correction

Unbiased estimation:

$$\hat{\mathbf{R}}_k^* = \hat{\mathbf{R}}_{k|k-1} \exp_m(\hat{\omega}_{k-1}) \tag{7}$$

Correction:

$$\hat{\mathbf{R}}_k = \hat{\mathbf{R}}_k^* \exp_m(\mathbf{K}_k(\mathbf{y}_k - h(\hat{\mathbf{R}}_k^*))) \tag{8}$$

$$\exp(\xi_k) = \hat{\mathbf{R}}_k^{-1}\mathbf{R}_k$$

$$\vdots$$

$$\xi_k = -\mathbf{M}_k\mathbf{v}_k + \mathbf{K}_k\mathbf{H}_k^*\mathbf{M}_k\mathbf{v}_k - \mathbf{K}_k\mathbf{v}_k \tag{9}$$

$E(\xi_k) = 0$, we need to find $\mathbf{K}_k = \arg\min tr(\mathbf{P}_k)$, where $\mathbf{P}_k = Cov(\xi_k)$

$$\mathbf{K}_k = -\mathbf{M}_k\mathcal{R}_k(\mathbf{I}_m - \mathbf{H}_k^*\mathbf{M}_k)^T \left((\mathbf{I}_m - \mathbf{H}_k^*\mathbf{M}_k)\mathcal{R}_k(\mathbf{I}_m - \mathbf{H}_k^*\mathbf{M}_k)^T\right)^\dagger \tag{10}$$

$$\mathbf{P}_k = \mathbf{K}_k(\mathbf{I}_m - \mathbf{H}_k^*\mathbf{M}_k)\mathcal{R}_k\mathbf{M}_k^T + \mathbf{M}_k\mathcal{R}_k\mathbf{M}_k^T \tag{11}$$

where $\mathbf{h}(\mathbf{R}_k) - \mathbf{h}(\hat{\mathbf{R}}_k^*) \approx \mathbf{H}_k^*\xi_{k|k-1}$

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$$\hat{\mathbf{R}}_k^* = \hat{\mathbf{R}}_{k|k-1} \exp_m(\hat{\omega}_{k-1}) \tag{7}$$

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where $\mathbf{h}(\mathbf{R}_k) - \mathbf{h}(\hat{\mathbf{R}}_k^*) \approx \mathbf{H}_k^*\xi_{k|k-1}$

UMV-SO(3)

Algorithm 1 UMV-SO(3)

Input: $\hat{\mathbf{R}}_{k-1}$, $\mathbf{P}_{k-1}$, $\mathbf{y}_k$

▷ **Biased Prediction:**

1: $\hat{\mathbf{R}}_{k|k-1} = \hat{\mathbf{R}}_{k-1}$

▷ **Unknown input estimation:**

2: $\mathbf{H}_k = \left.\dfrac{\partial h\left(\hat{\mathbf{R}}_{k|k-1}\exp_m(\xi)\right)}{\partial \xi}\right|_{\xi=0}$

3: $\tilde{\mathcal{R}}_k = \mathbf{H}_k\mathbf{P}_{k-1}\mathbf{H}_k^T + \mathbf{H}_k\mathcal{Q}_{k-1}\mathbf{H}_k^T + \mathcal{R}_k$

4: $\mathbf{M}_k = \left(\mathbf{H}_k^T\tilde{\mathcal{R}}_k^{-1}\mathbf{H}_k\right)^{-1}\mathbf{H}_k^T\tilde{\mathcal{R}}_k^{-1}$

5: $\hat{\omega}_{k-1} = \mathbf{M}_k(\mathbf{y}_k - h(\hat{\mathbf{R}}_{k|k-1}))$

▷ **Prior state estimation:**

6: $\hat{\mathbf{R}}_k^* = \hat{\mathbf{R}}_{k|k-1}\exp_m(\hat{\omega}_{k-1})$

▷ **Correction:**

7: $\mathbf{H}_k^* = \left.\dfrac{\partial h\left(\hat{\mathbf{R}}_k^*\exp_m(\xi)\right)}{\partial \xi}\right|_{\xi=0}$

8: $\mathcal{R}_k^* = (\mathbf{I}_m - \mathbf{H}_k^*\mathbf{M}_k)\mathcal{R}_k(\mathbf{I}_m - \mathbf{H}_k^*\mathbf{M}_k)^T$

$\mathcal{S}_k^* = \mathbf{M}_k\mathcal{R}_k(\mathbf{I}_m - \mathbf{H}_k^*\mathbf{M}_k)^T$

9: $\mathbf{K}_k = -\mathcal{S}_k^*\left(\mathcal{R}_k^*\right)^\dagger$

10: $\hat{\mathbf{R}}_k = \hat{\mathbf{R}}_k^*\exp_m(\mathbf{K}_k(\mathbf{y}_k - h(\hat{\mathbf{R}}_k^*)))$

11: $\mathbf{P}_k = \mathbf{K}_k\mathcal{S}_k^{*T} + \mathbf{M}_k\mathcal{R}_k\mathbf{M}_k^T$

12: **return** $\hat{\mathbf{R}}_k$, $\mathbf{P}_k$

Simulation Setup

Algorithm	Acc	Mag	Gyro	Current consumptions
UMV-SO(3)	✓	✓		Microampere range
TRIAD	✓	✓		Microampere range
IEKF	✓	✓	✓	Milliampere range
- Monte Carlo simulations with 100 runs for each scenario.
- We use the following true angular velocities.

True angular velocity (rad/s)	
ω_x	$2.0\cos(0.2\pi k\Delta T)$
ω_y	$1.5\cos(0.6\pi k\Delta T)$
ω_z	$1.0\cos(1.0\pi k\Delta T)$

- The accelerometer and magnetometer noises were set to be zero-mean white noise signals with standard deviations of $\sigma_a = 0.01\ m/s^2$, and $\sigma_m = 0.005\ Gauss$.

Compare with IEKF and TRIAD results

RMSE (in degrees) for both UMV-SO(3) and TRIAD

TRIAD RMSE (degrees)	**UMV-SO(3)** RMSE (degrees)
0.73	0.47

- UMV-SO(3) shows a **39%** improvement in RMSE compared to TRIAD.

RMSE (in degrees) for both UMV-SO(3) and IEKF for various gyroscope noise level

σ_ω (rad/sec)	**IEKF** RMSE (degrees)	**UMV-SO(3)** RMSE (degrees)
0.01	0.35	0.47
0.05	0.60	
0.10	0.71	

- UMV-SO(3) has comparable accuracy with IEKF.
- IEKF remains sensitive to the gyroscope noise.
- IEKF consumes more power than UMV-SO(3).
- IEKF needs to know the gyroscope noise covariance matrix and bias.

Conclusion

Results Summary:

- UMV-SO(3) provides an effective solution for attitude estimation.
- Monte Carlo simulations showed that our UMV-SO(3) algorithm outperformed TRIAD.
- UMV-SO(3) has comparable accuracy with IEKF.

Contributions Summary:

- Design algorithm for state estimation on SO(3) with unknown input without direct feedthrough to the output.
- The design of a novel algorithm for gyro-free attitude estimation.

G. Shaaban, H. Fourati, A. Kibangou and C. Prieur, "Attitude Estimation on SO(3) with Unknown Input" submitted to Automatica.

Related Contribution

- Design algorithm for state estimation on SO(3) with unknown input having direct feedthrough to the output.
  The design of a novel algorithm for attitude estimation using MARG sensors under unknown external acceleration. [Shaaban et al., 2023] [2]

[2] G. Shaaban, H. Fourati, A. Kibangou and C. Prieur, "MARG Sensor-Based Attitude Estimation on SO(3) Under Unknown External Acceleration," in IEEE Control Systems Letters, vol. 7, pp. 3795-3800, 2023.

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Thank You